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Negative modes in the four–dimensional stringy wormholes

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Abstract

We study the Giddings–Strominger wormholes in string theories. We found a non–singular wormhole solution and analyzed the perturbation around this wormhole solution. We have used the bilinear action to obtain Schrödinger–type equation for perturbation fields assuming a linear relation between the perturbation fields. With this analysis, we found an infinite number of negative modes among $O(4)$–symmetric fluctuations about the non–singular wormhole background.

Euclidean wormholes–solutions to the euclidean Einstein equations that connect two asymptotically flat regions–are considered as saddle points of the functional integral and are very important for semiclassical calculations of transition probabilities of topological change in quantum gravity. There are many kinds of euclidean wormhole solutions. In four–dimensions the following matters which support the throat of the wormhole were adopted: axion fields [1], scalar fields [2], SU(2) Yang–Mills fields [3]. Higher–dimensional wormhole solutions were obtained [4,5] and higher–derivative correction to the Einstein–Hilbert action was considered [6]. Recently, we found the D–wormhole solution in type IIB superstring theory [7]. However, it turned out to be a ten–dimensional singular wormhole with infinite euclidean action density.

On the other hand, we are interested in the contribution of wormhole configurations to the euclidean functional integral for the forward "flat space $\rightarrow$ flat space " amplitude. Rubakov and Shvedov [8] decided semiclassically whether Giddings–Strominger wormhole makes real or complex contributions into the functional integral in four–dimensional curved space . On the analogy of the analysis of instantons/bounces in the quantum field theory, it is found that the wormhole contribution is imaginary since there exists one negative mode ($\omega^2 = -4$) among fluctuations around the classical euclidean solution. This means that the classical solution with one negative mode is not stable against the fluctuations and thus belongs to the bounce.

In this paper, we study the Giddings–Strominger wormholes in string theories [9]. Hereafter we wish to call these as stringy wormholes to distinguish the previous Giddings–Strominger wormhole. We have both the singular wormhole as well as the non–singular one. It is carried out the analysis of $O(4)$–symmetric fluctuations about the non–singular wormhole background. We use the bilinear action to obtain Schrödinger–type equation for perturbation fields assuming a linear relation between perturbation fields. With this analysis, we find an infinite number of negative modes among $O(4)$–symmetric fluctuations about the non–singular wormhole background.

Our analysis is similiar to the stability analysis of the black holes [10], which is classical

solution in curved spacetime with the Minkowski signature. One easy way of understanding a black hole is to find out how it reacts to external perturbations. We can visualize the black hole as presenting an effective potential barrier (or well) to the on–coming waves. As a compact criterion for the black hole case, it is unstable if there exists a potential well to the on–coming waves. This is so because the Schrödinger–type equation with the potential well always allows the bound states as well as scattering states. The former shows up as an imaginary frequency mode ($\omega^2 < 0$), leading to an exponentially growing mode with time. If one finds any exponentially growing perturbation, the black hole turns out to be unstable.

Our starting action is the NS–NS sector of ten–dimensional string theory [9],

$$S_{10} = \int d^{10}x\sqrt{g_{10}}\; e^{\phi}[\,-R - (\nabla\phi)^2 + H^2], \tag{1}$$

where ϕ is the dilaton and $H = dB$ with a NS–NS two–form B. Here we do not consider the R–R sector for simplicity [11]. The ten–dimensional theory can be reduced to four–dimensional one by the compactification on a six–dimensional Calabi–Yau manifold. This is realized ($M^{10} \to M^4 \times M^6$) by giving the following vacuum expectation values:

$$\begin{aligned}
&\bar{g}_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & e^{D(x)/\sqrt{3}}g_{mn}(y) \end{pmatrix}, \\
&\bar{B}_{\mu\nu} = B_{\mu\nu}(x), \\
&\bar{B}_{mn} = (1/6)a(x)b_{mn}(y), \\
&\bar{\phi} = \phi(x).
\end{aligned} \tag{2}$$

and the rest of fields will be taken to zero. Here $\mu, \nu, \cdots (m, n, \cdots)$ denote four (six)-dimensional indices, and $x(y)$ represent four (six)-dimensional coordinates. The field equations for the graviton, dilaton, and two–form field are satisfied if the internal manifold (M^6) is Calabi–Yau (Ricci–flat and Kähler) and the equations of motion obtained from the four–dimensional effective action

$$S_4 = \int d^4x\sqrt{g}[\,-R + \frac{1}{2}(\nabla D)^2 + \frac{1}{2}e^{-\frac{2}{\sqrt{3}}D}(\nabla a)^2 + \frac{1}{2}(\nabla\Delta)^2 + e^{2\Delta}H^2], \tag{3}$$

where

$$\Delta \equiv \phi + \sqrt{3} D, \qquad g_{\mu\nu} = e^{\Delta} \tilde{g}_{\mu\nu}, \tag{4}$$

are satisfied.

Now let us find stringy wormhole solution by considering either a or $B_{\mu\nu}$ as matter which supports the throat of the wormhole. Here we confine our main interest to the first case (the non–singular wormhole). The latter case leads to the singular wormhole. The non–singular case is realized when $H = \Delta = 0$. The action is given by

$$S = \int d^4x \sqrt{g} [- R + \frac{1}{2} (\nabla D)^2 + \frac{1}{2} e^{-\frac{2}{\sqrt{3}} D} (\nabla a)^2]. \tag{5}$$

One can consider $a(x)$ as the source of the wormhole. We thus take the Noether current $J_\mu = e^{-(2/\sqrt{3})D} \partial_\mu a$ and require its conservation

$$\partial_\mu (\sqrt{g} J^\mu) = 0. \tag{6}$$

Therefore we have to perform the functional integration over conserved current densities. We introduce the general $O(4)$–symmetric euclidean metric as

$$ds^2 = N^2(\rho) d\rho^2 + R^2(\rho) d\Omega_3^2 \tag{7}$$

with two scale factors (N, R). The $O(4)$–symmetric current density has one non–zero component $(J^0(\rho))$ and its conservation in (6) means that $\sqrt{g} J^0$ is a constant. This constant is related to the global charge Q of the wormhole $(Q/Vol(S^3))$. Thus one finds

$$J^0 = \frac{Q}{2\pi^2} \frac{1}{N R^3}. \tag{8}$$

The action (5) can be rewritten as

$$S = 6 \int d^4x [- \frac{R R'^2}{N} - N R + \frac{1}{12} \frac{R^3}{N} D'^2 + \frac{Q^2}{48\pi^4} \frac{N}{R^3} e^{\frac{2}{\sqrt{3}} D}], \tag{9}$$

where the prime means the derivative with respect to ρ. From the above action, the equations of motion are

$$\frac{RR'^2}{N^2} - R - \frac{1}{12}\frac{R^3}{N^2}D'^2 + \frac{Q^2}{48\pi^4}\frac{1}{R^3}e^{\frac{2}{\sqrt{3}}D} = 0, \tag{10}$$

$$-\frac{R'^2}{N} + 2(\frac{RR'}{N})' - N + \frac{1}{4}\frac{R^2}{N}D'^2 - \frac{Q^2}{16\pi^4}\frac{N}{R^4}e^{\frac{2}{\sqrt{3}}D} = 0, \tag{11}$$

$$-\frac{1}{6}(\frac{R^3D'}{N})' + \frac{Q^2}{24\sqrt{3}\pi^4}\frac{1}{NR^3}e^{\frac{2}{\sqrt{3}}D} = 0. \tag{12}$$

For $N = 1$ gauge, (10) and (12) are reduced to

$$R'^2 = 1 + \frac{1}{12}R^2D'^2 - \frac{Q^2}{48\pi^4}\frac{1}{R^4}e^{\frac{2}{\sqrt{3}}D}, \tag{13}$$

$$(R^3D')' = \frac{Q^2}{4\sqrt{3}\pi^4}\frac{1}{R^3}e^{\frac{2}{\sqrt{3}}D}. \tag{14}$$

From (14), one finds the dilaton equation

$$R^6D'^2 = \frac{Q^2}{4\pi^4}e^{\frac{2}{\sqrt{3}}D} - \frac{Q^2}{4\pi^4}e^{\frac{2}{\sqrt{3}}D_0}, \tag{15}$$

where the integration constant is chosen so that D has vanishing derivative at the wormhole neck ($\rho = 0$). Substituting this into (13), one obtains

$$R'^2 = 1 - \frac{R_0^4}{R^4}, \qquad R_0^4 = \frac{Q^2}{48\pi^4}e^{\frac{2}{\sqrt{3}}D_0}. \tag{16}$$

Here $R_0 = R(\rho = 0)$ corresponds to the radius of wormhole neck ($R' = 0$). Equation (11) is satisfied with (15) and (16) and thus is a redundant one. The resulting solution (stringy wormhole) to (16) has the asymptotic behavior $R(\rho) \to \pm\rho$ as $\rho \to \pm\infty$, corresponding to two asymptotically flat regions and has minimum at R_0. Further (15) is solved to obtain

$$e^{-\frac{2}{\sqrt{3}}D} = \frac{Q^2}{48\pi^4}\frac{1}{R^4}, \tag{17}$$

which will prove very useful for the computation of the perturbed action on later. Note that D is non–singular for finite ρ and thus the integrand of the action is finite too. For an explicit calculation, we wish to solve the differential equation (16) by numerical analysis. We introduce the rescalings $(\rho/\rho_0, R/R_0, D/D_0)$ with $\rho_0 = R_0$. The resulting solution is

shown in Fig.1. Far from the wormhole throat ($\rho/R_0 > 1$), one can ignore the effect of gravity and the euclidean space becomes flat ($R \sim \rho$). Here one can find the wormhole neck ($R' = 0$) near $\rho = 0$. Now let us substitute the results of $R(\rho)/R_0$ in Fig.1 into (17). Then one obtains the behavior of the wormhole dilaton ($D(\rho)$). As is shown in Fig.2, D does not have any singular point.

Let us now consider $O(4)$–symmetric fluctuations about the non–singular wormhole solution. In general, the interpretation of the wormhole depends on whether or not there are negative modes around the solution. If one finds odd number of negative modes, the solution corresponds to a bounce and describes the nucleation and growth of wormhole in the Minkowski spacetime. If there are even number of negative modes, the path integral would be real and classical solution would resemble an instanton rather than a bounce. If there is no negative mode, the solution is called an instanton and describes the tunneling and mixing of two states of the same energy. The small fluctuations are given by

$$R(\rho) = R_c(\rho) + r(\rho), \quad N(\rho) = 1 + n(\rho), \quad D(\rho) = D_c(\rho) + d(\rho), \tag{18}$$

where R_c, D_c represent the classical wormhole background. Substituting these into (9) and then take only the bilinear parts in (r, n, d) of the action. This is because from this part one can derive the linearized equations which are essential for the fluctuation study. Here we choose the $n(\rho) = 0$ gauge, since the quadratic action is invariant under the $O(4)$–general coordinate transformations. The bilinear action is then given by

$$\begin{aligned} S_{bil} = 12\pi^2 \int d\rho \Big[& - R_c r'^2 - 2R_c' r r' + \frac{1}{12}(R^3 d'^2 + 6R_c^2 D_c' r d' + 3R_c D_c'^2 r^2) \\ & + \frac{Q^2}{48\pi^4}\frac{1}{R_c^3} e^{\frac{2}{\sqrt{3}}D} (6\frac{r^2}{R_c^2} - 2\sqrt{3}\frac{dr}{R_c} + \frac{2}{3}d^2)\Big]. \end{aligned} \tag{19}$$

After some calculation, (19) can be rewritten as

$$\begin{aligned} S_{bil} = 12\pi^2 \int d\rho \Big[& - R_c r'^2 + (\frac{9}{R_c} - \frac{R_0^4}{R_c^5}) r^2 + \frac{R_c^3}{12} d'^2 \\ & + \frac{1}{2} R_c^2 D_c' r d' - 2\sqrt{3} dr + \frac{2}{3} R_c d^2 \Big] \end{aligned} \tag{20}$$

with the boundary terms which are not relevant for our study. One confronts with difficulty in dealing with (20). This is because of the presence of r-d coupling terms. Actually one has

to find the new canonical variables that diagonalize the action (20). However, thanks to the relation (17), one has the relation between r and d. Linearizing (17) leads to $d = 2\sqrt{3}r/R_c$ and inserting this into (20), we obtain $S_{bil} = 0$ which leads to a trivial case. In order to avoid this trivial case, we assume the relation as

$$d = 2\sqrt{3}\alpha \frac{r}{R_c} \tag{21}$$

by introducing α as the parameter. This means that d is not an independent variable. The above is the simplest assumption which is appropriate in the spirit of linear perturbation. Otherwise, the analysis becomes very difficult. Using (21), we find the desirable bilinear form

$$S_{bil} = 12\pi^2(\alpha^2 - 1)\int d\rho \Big[R_c r'^2 + (\frac{R_0^4}{R_c^5} + \frac{\alpha-1}{\alpha+1}\frac{9}{R_c})r^2\Big]. \tag{22}$$

One can easily check that $S_{bil} = 0$ for $\alpha = 1$. Since the bilinear from (22) is positive definite for $\alpha^2 > 1$, there is no negative modes in this region. Thus the range of the parameter should be confined to $\alpha^2 < 1$. But for $\alpha^2 < 1$, the action is unbounded from below, because of the negative sign of the kinetic term. In this case, we need the GHP rotation [12] for scale factor ($r \to ir$). Taking the variation of the action (22) with respect to r, on gets the Schrödinger–type equation

$$R_c\Big[-(R_c r')' + (\frac{R_0^4}{R_c^5} + \frac{\alpha-1}{\alpha+1}\frac{9}{R_c})r\Big] = \omega^2 r. \tag{23}$$

Here we choose a prefactor R_c on the left–hand side in such a way that the above equation can be solved explicitly. From now on we are interested in negative mode $\omega^2 = -|\omega|^2$. For simplicity we set $R_0 = 1$. Introducing a new variable $y = R_c^{-4}$ and rewriting $r = R_c^{-p}\psi(y)$, (23) is reduced to the form of a hypergeometric equation

$$y(1-y)\frac{d^2\psi}{dy^2} + \{(1+\frac{p}{2}) - (\frac{3}{2}+\frac{p}{2})y\}\frac{d\psi}{dy} - \frac{1}{16}(p+1)^2\psi = 0, \tag{24}$$

with

$$p^2 = |\omega|^2 + 9\frac{\alpha-1}{\alpha+1}. \tag{25}$$

Note that the variables y and ρ are not in one to one correspondence. This can be cured by requiring that $r(\rho)$ is either symmetric or antisymmetric with respect to ρ. One finds that y and $1-y$ are symmetric, while $\sqrt{1-y}$ is antisymmetric in ρ. The symmetric solution of (24) is

$$\psi(y) = C_1 F(\frac{p+1}{4}, \frac{p+1}{4}; 1+\frac{p}{2}; y) + C_2 F(\frac{p+1}{4}, \frac{p+1}{4}; \frac{1}{2}; 1-y), \tag{26}$$

where C_1 and C_2 are arbitrary constants. Further one requires the small perturbation such that $r(\rho)$ be finite at both ends ($\rho \to \infty (y \to 0)$ and $\rho \to 0 (y \to 1)$). The last term in (26) behaves $y^{-(p+1)^2/8}$ as $y \to 0$, and this in turn gives us $r_{y\to 0} \to y^{-(p^2+1)/8}$. This diverges as $y \to 0$ and thus we set $C_2 = 0$. On the other hand, the first term in (26) is finite at both ends. We also impose that the eigenfunction r is square integrable with the weight $d\rho/R_c(\rho)$. This is compatible with the choice of the prefactor R_c in (23). The integrability condition is realized as

$$\int_0^\infty \frac{r^2 d\rho}{R_c} = \int_0^1 \psi^2 y^{p/2-1}(1-y)^{1/2-1} dy < M \int_0^1 y^{p/2-1}(1-y)^{1/2-1} dy = MB(p/2, 1/2), \tag{27}$$

where M is the maximum value of ψ^2 in [0,1] and $B(p/2, 1/2) = \Gamma(p/2)\Gamma(1/2)/\Gamma(p/2+1/2)$ is the beta–function. The condition for finite $B(p/2, 1/2)$ requires that p should be positive. From (25), this condition is satisfied if $|\omega|^2 > -9(\alpha-1)/(\alpha+1)$ for $\alpha^2 < 1$. Hence one can always find negative modes for all positive p.

We perform the analysis of $O(4)$–symmetric fluctuations on the stringy wormhole background with the gauge $n(\rho) = 0, r(\rho) \neq 0$. Instead of diagonalizing the quardratic action, we choose the relation $d = 2\sqrt{3}\alpha r/R_c$ which is inspired by (17). Rubakov and Shvedov [8] reported that there exists only one negative mode $r^{(-)}(\rho) = 1/R_c^2(\rho)$ with $\omega^2 = -4$ for pure gravity case. The existence of one negative mode implies that the wormhole contribution into the functional integral is imaginary, which corresponds to the instability of the parent universe against the emission of a baby universe. In our case $r(\rho) = R_c^{-p}$ with $p = -1$ satisfies (23) over the entire region. But this solution is not a small perturbation and thus we discard it. On the other hand, we find a continuous spectrum of negative modes for

positive p by requiring both symmetric property and integrability. This diffrence comes from the fact that the last term of (24) is different from (8) in Ref. [8]. It has been shown by Coleman [14] that the bounce interpretation of a classical solution requires exactly one negative mode. In general, the reality and imaginarity of the path integral depends on the sign of the determinant of fluctuatuons. The essential property is whether the number of negative modes is odd or even. The odd case belongs to the bounce, while the even case is related to the instanton. Here we obtain the continuous spectrum of negative modes. The existence of an infinite number of negative modes leads to diffrent problems. Lavrelashvili, Rubakov and Tinyakov(LRT) [15] pointed out that an infinite number of negative modes may appear around the bounce. But Tanaka and Sasaki [16] argued that the above LRT claim is an artifact due to their inadequate choice of gauge (LRT gauge), which was inevitably implied by the Lagrangian formalism. For the LRT gauge of $n(\rho) \neq 0, r(\rho) = 0$ in [16], one can obtain the bilinear action from (9). One has to use the constraint equation (10) to eliminate the $n(\rho)$–terms. Unfortunately, we cannot get the relation between $n(\rho)$ and $d(\rho)$ by linearizing (10). Further the corresponding action turns out to be trivial.

In our case, we choose the gauge of $n(\rho) = 0, r(\rho) \neq 0$. Under this gauge, one has to perform the Hamiltonian analysis arisen from Ref. [16]. At this stage, it is not clear to conclude whether the stringy wormhole is a bounce or an instanton.

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FIGURES

Fig. 1: R/R_0 as a function of ρ/R_0. The solid, dotted and dashed lines correspond to wormhole scale factor (R/R_0), $R(\rho)/R_0 \approx 1.274$, and $R/R_0 = \rho$. The singular point (ρ_{sg}) is determined as a solution to $R(\rho_{sg})/R_0 = 1/\sqrt{\cos(\pi/2\sqrt{3})} \approx 1.274$.

Fig.2: D/D_0 as a function of ρ/R_0. No singular point is found.

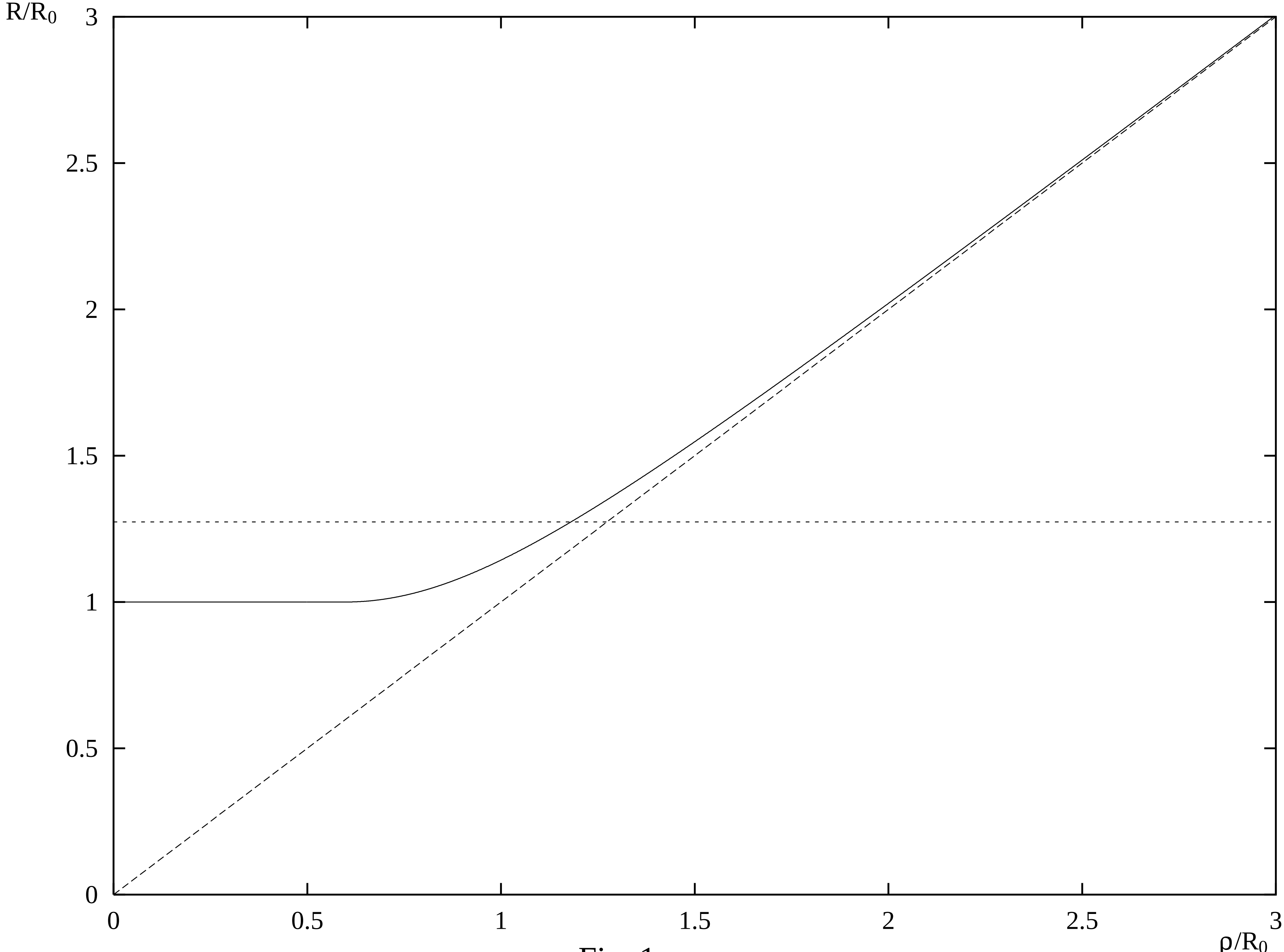

Fig. 1

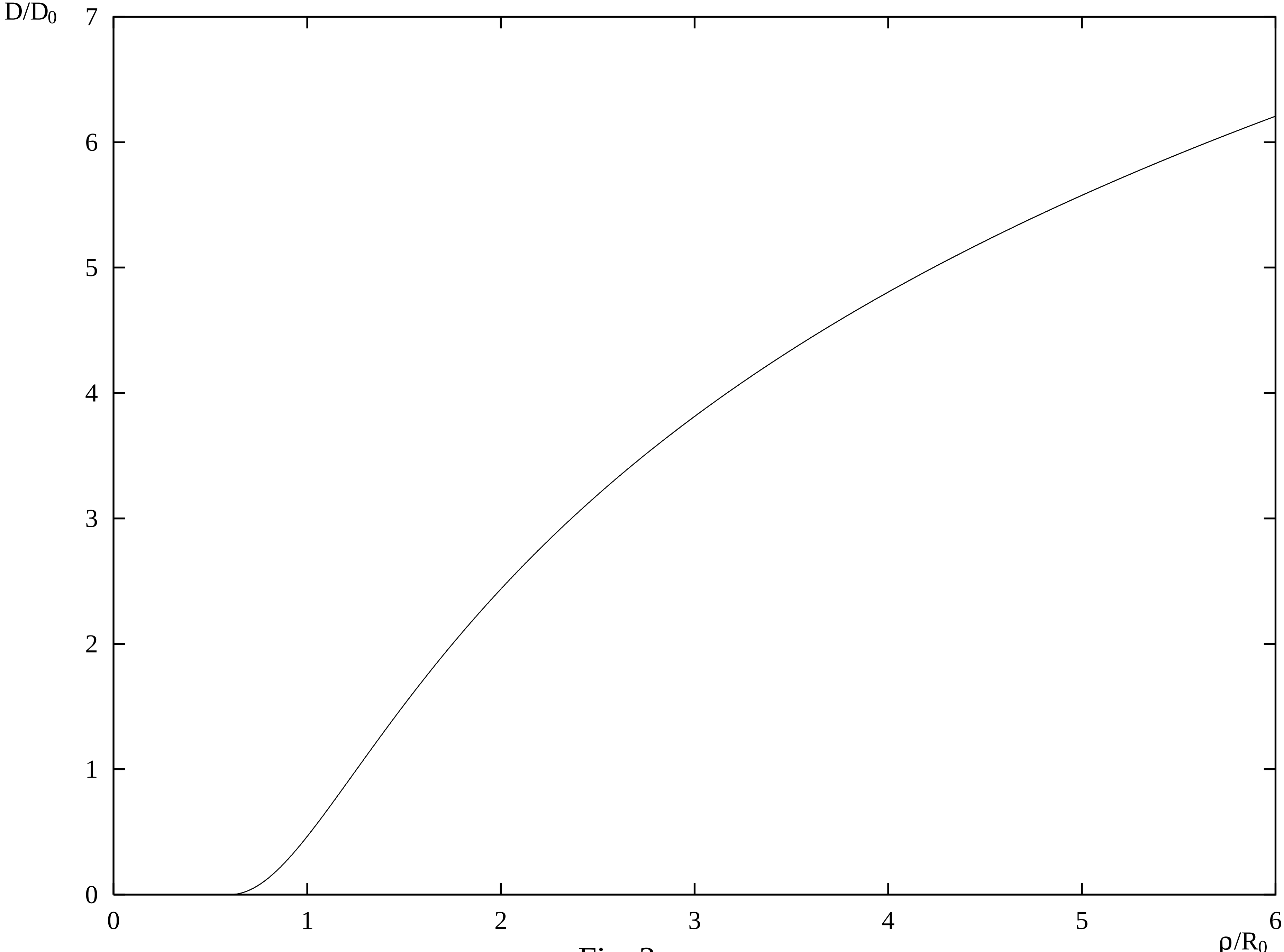

Fig. 2